Math 10A with Professor Stankova
Worksheet, Discussion #33; Monday, 11/13/2017
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Standard Deviation

Example

1. Let $f(x) = e \cdot e^x$ for $x \leq -1$ and 0 otherwise. Find the standard deviation of this distribution.

Solution: First we need to find the mean of this distribution. The mean is

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} x(e \cdot e^x)dx + \int_{-1}^{\infty} 0dx = e\int_{-\infty}^{-1} xe^x dx$$

$$= e(xe^x - e^x|_{-\infty}^{-1}) = e[(-e^{-1} - e^{-1}) - 0] = -2.$$

To find the standard deviation, we first find the variance and then take the square root. There are two ways to do this, the latter is a bit easier

$$\sigma^2 = \int_{-\infty}^{\infty} (x - (-2))^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{-\infty}^{-1} x^2(e \cdot e^x)dx - 4$$

$$= e(x^2e^x - 2xe^x + 2e^x|_{-\infty}^{-1}) - 4 = e(e^{-1} + 2e^{-1} + 2e^{-1}) - 4 = 5 - 4 = 1.$$

So the standard deviation is $\sigma = 1$.

2. Find the standard deviation of the set $\{1, 2, 3\}$.

Solution: First we find the mean, which is the average, and hence it is $\frac{1+2+3}{3} = 2$. Then we calculate the variance which is

$$\frac{(1-2)^2 + (2-2)^2 + (3-2)^2}{3} = \frac{2}{3}.$$

For statisticians, or something called an unbiased estimator, we divide by $N - 1$ or 2 instead to get a variance of 1. The standard deviation is thus $\sqrt{6}/3$ or 1.

Problems

3. True **FALSE** The standard deviation always exists.

Solution: The standard deviation requires the mean to exist, and sometimes that doesn't exist.

4. True **FALSE** Sometimes, we take the standard deviation to be the negative square root of the variance.

Solution: The standard deviation is always nonnegative.

5. **TRUE** False The variance is always nonnegative.

Solution: The variance is $\int (x-\mu)^2 f(x)dx$ and both $(x-\mu)^2 \geq 0$ and $f(x) \geq 0$ so $(x-\mu)^2 f(x) \geq 0$ so the integral must be nonnegative too.

6. **TRUE** False If the mean doesn't exist, then the standard deviation doesn't exist.

Solution: The formula for the standard deviation requires the mean, so if the mean doesn't exist, then we can't talk about the standard deviation.

7. True **FALSE** If the mean exists, then the standard deviation exists.

Solution: It is possible for the mean to exist but the standard deviation not to exist. For example, the distribution $\frac{1}{x^3}$ on $x \geq 1$ has the mean existing but the standard deviation not.

8. Let $f(x)$ be $2/3x$ from $1 \leq x \leq 2$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_1^2 2/3x^2 dx = \frac{2}{9}x^3|_1^2 = \frac{14}{9}.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-14/9)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_1^2 2/3x^3 dx - (14/9)^2 = \frac{5}{2} - \frac{196}{81} = \frac{13}{162}.$$

Thus, $\sigma = \sqrt{13/162} = \sqrt{26}/18$.

9. Let $f(x)$ be $-4/x^5$ for $x \leq -1$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} -4/x^4 dx = \frac{4}{3}x^{-3}|_{-\infty}^{-1} = \frac{-4}{3}.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-(-4/3))^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{-\infty}^{-1} -4/x^3 dx - (-4/3)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

Thus, $\sigma = \sqrt{2/9} = \sqrt{2}/3$.

10. Let $f(x)$ be the uniform distribution on $0 \leq x \leq 10$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: Since f is the uniform distribution on $[0, 10]$, we know that $f(x) = \frac{1}{10-0} = \frac{1}{10}$ on $[0, 10]$ and 0 everywhere else. First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{10} x/10 dx = x^2/20|_0^{10} = 5.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-5)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{0}^{10} x^2/10 dx - 5^2 = \frac{100}{3} - 25 = \frac{25}{3}.$$

Thus, $\sigma = \sqrt{25/3} = 5\sqrt{3}/3$.

11. Let $f(x)$ be $-2x$ from $-1 \leq x \leq 0$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{0} -2x^2 dx = -2/3x^3|_{-1}^{0} = \frac{-2}{3}.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-(-2/3))^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{-1}^{0} -2x^3 dx - (-2/3)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Thus, $\sigma = \sqrt{1/18} = \sqrt{2}/6$.

12. Let $f(x)$ be $24/x^4$ for $x \geq 2$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{2}^{\infty} 24/x^3 dx = -12x^{-2}|_2^{\infty} = 3.$$

Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-3)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{2}^{\infty} 24/x^2 dx - 3^2 = 12 - 9 = 3.$$

Thus, $\sigma = \sqrt{3}$.

13. Let $f(x)$ be the uniform distribution on $-20 \leq x \leq -10$ and 0 everywhere else. Find the standard deviation of this distribution.

Solution: Since f is the uniform distribution on $[-20, -10]$, we know that $f(x) = \frac{1}{-10-(-20)} = \frac{1}{10}$ on $[-20, -10]$ and 0 everywhere else. First we find the mean as

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-20}^{-10} x/10 dx = x^2/20|_{-20}^{-10} = -15.$$

This could also be found by noting that the distribution is symmetric around -15 and that it is finite. Then, to find the variance, we take

$$\sigma^2 = \int_{-\infty}^{\infty} (x-(-15))^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{-20}^{-10} x^2/10 dx - (-15)^2 = \frac{700}{3} - 225 = \frac{25}{3}.$$

Thus, $\sigma = \sqrt{25/3} = 5\sqrt{3}/3$.

Chebyshev's Inequality

Example

14. Let $f(x) = e \cdot e^x$ for $x \leq -1$ and 0 otherwise. Estimate the probability $P(-4 \leq X \leq 0)$. For what a can we say that $P(X \geq a) \geq 0.99$?

Solution: Since the mean is -2 and the standard deviation is 1, using Chebyshev's inequality, we have that $P(-4 \leq X \leq 0) = P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$. So an estimate would be $\frac{3}{4}$. The real answer is ≈ 0.95.

In order for $P(X \geq a) \geq 0.99$, we set $a = \mu - k\sigma$ and we know that $P(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2}$. We want to set this lower bound to 0.99 and doing so gives $k = 10$. Thus, we have that $P(\mu - 10\sigma \leq X \leq \mu + 10\sigma) = P(-12 \leq X \leq 8) = P(-12 \leq X) \geq 0.99$. So, we have that $a = -12$.

Problems

15. True **FALSE** Chebyshev's inequality can tell us what the probability actually is.

Solution: Like error bounds, Chebyshev's inequality just gives us an estimate and not the actual probability.

16. True **FALSE** For Chebyshev's inequality, the k must be an integer.

Solution: We can take k to be any positive real number.

17. True **FALSE** Chebyshev's inequality can help us estimate $P(\mu - \sigma \leq X \leq \mu + \sigma)$.

Solution: Using Chebyshev's inequality, we get that this probability is greater than $1 - 1/1 = 0$ which we knew anyway because it is a probability.

18. Let $f(x)$ be $2/3x$ from $1 \leq x \leq 2$ and 0 everywhere else. Estimate $P(10/9 \leq X \leq 2)$.

Solution: The mean is 14/9 and so this probability is $P(14/9 - 4/9 \leq X \leq 14/9 + 4/9)$. Letting $4/9 = k\sigma = k\sqrt{26}18$, we calculate that $k = \frac{8}{\sqrt{26}}$. Then, we have that $P(10/9 \leq X \leq 2) \geq 1 - 1/k^2 = 1 - 1/(64/26) = \frac{19}{32}$.

19. Let $f(x)$ be $-4/x^5$ for $x \leq -1$ and 0 everywhere else. Estimate $P(X \geq -3)$

Solution: The mean is $-4/3$ and since $f(x) = 0$ for all $x > -1$, we have that $P(X \geq -2) = P(-4/3 - 5/3 \leq X \leq -4/3 + 5/3) \geq 1 - 1/k^2$. Here, we have that $5/3 = k\sigma = k\sqrt{2}/3$ and so $k = 5/\sqrt{2}$ and $1 - 1/k^2 = 1 - 1/(25/2) = \frac{23}{25}$.

20. Let $f(x)$ be the uniform distribution on $0 \leq x \leq 10$ and 0 everywhere else. Estimate $P(2 \leq X \leq 8)$.

Solution: The mean is 5 and so this probability is $P(5 - 3 \leq X \leq 5 + 3)$. Letting $3 = k\sigma = k5\sqrt{3}/3$, we calculate that $k = \frac{9}{5\sqrt{3}}$. Then, we have that $P(2 \leq X \leq 8) \geq 1 - 1/k^2 = 1 - 1/(81/75) = \frac{2}{27}$.

21. Let $f(x)$ be $-2x$ from $-1 \leq x \leq 0$ and 0 everywhere else. Estimate $P(-1 \leq X \leq -1/3)$.

Solution: First we know that $P(-1 \leq X \leq -1/3) = P(-2/3 - 1/3 \leq X \leq -2/3 + 1/3) \geq 1 - \frac{1}{k^2}$. Let $1/3 = k\sigma = k/\sqrt{18}$. Then $k = \frac{\sqrt{18}}{3} = \sqrt{2}$ and $1 - 1/k^2 = 1 - 1/2 = 1/2$.

22. Let $f(x)$ be $24/x^4$ for $x \geq 2$ and 0 everywhere else. Estimate $P(X \leq 5)$.

Solution: The mean is 3 and so $P(X \leq 5) = P(X \leq 3+2) = P(3-2 \leq X \leq 3+2)$ because $P(X \leq 1) = 0$. Let $2 = k\sigma = k\sqrt{3}$ so $k = 2/\sqrt{3}$. Then $1 - 1/k^2 = 1 - 1/(4/3) = \frac{1}{4}$.

23. Let $f(x)$ be the uniform distribution on $-20 \leq x \leq -10$ and 0 everywhere else. Estimate $P(-18 \leq X \leq -12)$.

Solution: The mean is -15 and so this probability is $P(-15 - 3 \leq X \leq -15 + 3)$. Letting $3 = k\sigma = k5\sqrt{3}/3$, we calculate that $k = \frac{9}{5\sqrt{3}}$. Then, we have that $P(2 \leq X \leq 8) \geq 1 - 1/k^2 = 1 - 1/(81/75) = \frac{2}{27}$.